Math 2E Selected Problems for the Final

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These are the problems out of the textbook that I listed as more theoretical.

Here's also some study tips:
1) Make sure you know the definitions of all the different integrals and objects. Mainly, understand all the bold faced words and boxed equations in the sections we've covered.
2) Make sure you understand your past homework problems, quizzes, and the practice final.
3) Understand the reasoning behind most of the extra theoretical problems.
4) A nice way to recap a lot of these things is to go through the Review sections in Chapter 15,16. I recommend reading the relevant concept check questions and doing the true false questions, too:
Chapter 15: Pg. 1061.
Chapter 16: Pg. 1148.

Disclaimer: The following solutions are only very sketched out! Some aren't even solutions, but just ideas or hints for the solutions.

16.3 Fundamental Theorem of Line Integrals

#29

This is basically that if a vector field is conservative, then $\nabla \times \mathbf{F} = 0$.

#30

This integral results from taking $\int_C \mathbf{F} \cdot \mathbf{dr}$ where $\mathbf{F}(x, y, z) = < y,\ x,\ xyz >$. The curl of this is not zero,

$$\nabla \times \mathbf{F} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & x & xyz \end{bmatrix} = (xz - 0)\hat{i} + (0 - yz)\hat{j} + (1 - 1)\hat{k}$$

where the first two components are not zero.

#35

a) The partials are

$$P_y = \frac{-(x^2 + y^2) + y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad Q_x = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

so we see they equal.

b) This integral is not path independent if the curve circles the origin. We see that if C_1 is the upper half of the circle (of radius 1), then we parameterize C_1 as $r_1(t) = < \cos t, \sin t >$ with $0 \leq t \leq \pi$, so the tangent is $r_1'(t) = < -\sin t, \cos t >$ and

$$\int_{C_1} \mathbf{F} \cdot \mathbf{dr} = \int_0^\pi \mathbf{F}(\mathbf{r_1}(\mathbf{t})) \cdot \mathbf{r_1'}(\mathbf{t}) dt = \int_0^\pi \frac{+\sin^2 t + \cos^2 t}{\cos^2 t + \sin^2 t} dt = \pi.$$

But, for C_2, the lower half of the circle, we parameterize it as the same $r_2(t) = <\cos t, -\sin t>$ from $0 \le t \le \pi$, so the tangent is $r_2'(t) = <-\sin t, -\cos t>$. Then,

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi \mathbf{F}(\mathbf{r_2(t)})\cdot \mathbf{r_2'(t)}dt = \int_0^\pi \frac{(-1)^3\sin^2 t - \cos^2 t}{\cos^2 t + \sin^2 t}dt = \int_0^\pi (-\sin^2 t - \cos^2 t)dt$$

$$= \int_0^\pi -1dt = -\pi.$$

We see that the value of these two integrals is different, so the integral is NOT path independent.

This does not violate anything because the components of $\mathbf{F}$, and their partials too, are not even continuous at the origin (0,0). The domain of $\mathbb{R}^2$ without the origin is not simply connected, and hence this non-simply-connectedness violates Theorem 6 (on Pg 1079).

16.4 Green's Theorem

#21

a) First we need to parameterize the line. It would be given by

$$r(t) = (1-t) < x_1,\ y_1 > +t < x_2,\ y_2 > = < x_1 + t(x_2 - x_1),\ y_1 + t(y_2 - y_1) >, \quad 0 \le t \le 1.$$

Then, $dx = x'(t)dt = (x_2 - x_1)dt$ and $dy = y'(t)dt = (y_2 - y_1)dt$. Then,

$$\int_C xdy - ydx = \int_0^1 (x_1 + t(x_2 - x_1))(y_2 - y_1)dt - \int_0^1 (y_1 + t(y_2 - y_1))(x_2 - x_1)dt$$

$$= x_1(y_2 - y_1) + \frac{(x_2 - x_1)(y_2 - y_1)}{2} - y_1(x_2 - x_1) - \frac{(y_2 - y_1)(x_2 - x_1)}{2}$$

$$= x_1(y_2 - y_1) - y_1(x_2 - x_1) = x_1y_2 - x_1y_1 - y_1x_2 + y_1x_1$$

$$= x_1y_2 - x_2y_1.$$

b) The boundary of the polygon is precisely the line segments connecting each (x_k, y_k) to the next (x_{k+1}, y_{k+1}) vertex - call each line segment L_k, with the last line segment L_n connecting (x_n, y_n) to (x_1, y_1). Recall as a consequence from Green's Theorem that

$$Area(Domain\ D) = \iint_D dxdy = \oint_{\partial D} \frac{x}{2}dy - \frac{y}{2}dx$$

since with $P = -y/2, Q = x/2$, then $Q_x - P_y = 1$. In particular, the boundary of D is precisely the union of each line segment, in that

$$\oint_{\partial D} = \int_{L_1} + \int_{L_2} + ... + \int_{L_{n-1}} + \int_{L_n} = \sum_{k=1}^n \int_{L_k}$$

so when we compute the integral,

$$Area(D) = \iint_D dxdy = \oint_{\partial D} \frac{x}{2}dy - \frac{y}{2}dx = \sum_{k=1}^n \frac{1}{2}\int_{L_k} xdy - ydx$$

where applying part (a) to each $\int_{L_k}$,

$$= \frac{1}{2}\sum_{k=1}^{n} x_k y_{k+1} - x_{k+1} y_k$$

where if we write this out,

$$= \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + ... + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)].$$

c) Is a computation - use the formula with just $n = 5$ vertices.

$$A = \frac{1}{2}[(0-0) + (6-1) + (2-0) + (0-(-2)) + (0-0)] = \frac{1}{2}[5+2+2] = 9/2.$$

#29

Here, $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \frac{<-y,\ x>}{x^2+y^2}$. Recall that from 16.3.35, we saw that its components $P = -\frac{y}{x^2+y^2}$ and $Q = \frac{x}{x^2+y^2}$ satisfied $Q_x = P_y$. Moreover, these components are smooth everywhere *except at the origin*. This is half of what we need for path independence. The main idea is that we do not enclose the origin. For any simple closed path that does not enclose the origin, we can enclose that curve in some region D of $\mathbb{R}^2$ that is simply connected. This means that $\mathbf{F}$ is conservative on D and since the curve lies in D, we can use the Fundamental theorem of line integrals. Since $\mathbf{F}$ is conservative, it is the gradient of some f on D, in that $\mathbf{F} = \nabla f$ on D. The curve is closed, so the final and initial point, call it P are the same. Then, $\oint_C \mathbf{F}\cdot\mathbf{dr} \overset{Fund.Thm}{=} f(P) - f(P) = 0$.

#31

See the practice final Problem 6.

16.5 Curl and Divergence

#19

Firstly, since $\nabla \bullet \nabla \times \mathbf{G} = 0$ always, check it:

$$\nabla \bullet \nabla \times \mathbf{G} = \nabla \bullet < x\sin y, \cos y, z - xy > = \sin y - \sin y + 1 \neq 0.$$

This doesn't even satisfy that $\nabla \bullet \nabla \times \mathbf{G} = 0$ so such a vector field $\mathbf{G}$ will not exist.

#20

This time, doing the same thing,

$$\nabla \bullet \nabla \times \mathbf{G} = \nabla \bullet < xyz, -y^2z, yz^2 > = yz - 2yz + 2yz \neq 0$$

again so such a vector field $\mathbf{G}$ again will not exist....

#21

See the practice final Problem 2(b). Irrotational means that the curl is zero.

#22

Incompressible means that the divergence is zero. But here,

$$\nabla \bullet \mathbf{F} = \partial_x f(y,z) + \partial_y g(x,z) + \partial_z h(x,y) = 0$$

since each component is independent of the variable that it is being differentiated with respect to.

#26

See practice final Problem 5(b).

Other problems like 23-29 are of this flavor too, but very long :(I'm not going to do them.

#30-32

Since the professor has had a few problems using the $\mathbf{r}$ vector, it may be nice to look at these again. For example, for 32,

32) if $\mathbf{F} = \mathbf{r}/r^p$, find $\nabla \cdot \mathbf{F}$. Is there a value of p such that the divergence is 0?

Here, first rewriting in Cartesian and following the definition of $\mathbf{r}$,

$$\mathbf{F}(x,y,z) = \frac{< x, y, z >}{(x^2+y^2+z^2)^{p/2}}.$$

The divergence is then (careful with factors of 2 and 1/2 cancelling)

$$\nabla \cdot \mathbf{F} = \frac{(x^2+y^2+z^2)^{p/2} - px^2(x^2+y^2+z^2)^{p/2-1}}{(x^2+y^2+z^2)^p} \quad \text{from quotient rule on } \partial/\partial x$$

$$+\frac{(x^2+y^2+z^2)^{p/2} - py^2(x^2+y^2+z^2)^{p/2-1}}{(x^2+y^2+z^2)^p} \quad \text{from quotient rule on } \partial/\partial y$$

$$+\frac{(x^2+y^2+z^2)^{p/2} - pz^2(x^2+y^2+z^2)^{p/2-1}}{(x^2+y^2+z^2)^p} \quad \text{from quotient rule on } \partial/\partial z$$

$$= \frac{3}{(x^2+y^2+z^2)^{p/2}} - \frac{p(x^2+y^2+z^2)(x^2+y^2+z^2)^{p/2-1}}{(x^2+y^2+z^2)^p}$$

$$= \frac{3}{(x^2+y^2+z^2)^{p/2}} - \frac{p(x^2+y^2+z^2)^{p/2}}{(x^2+y^2+z^2)^p} = \frac{3}{(x^2+y^2+z^2)^{p/2}} - \frac{p}{(x^2+y^2+z^2)^{p/2}}$$

so thus

$$\nabla \cdot \mathbf{F} = \frac{3-p}{(x^2+y^2+z^2)^{p/2}}.$$

Consequently, if $p = 3$, this is zero.

33

To begin the proof using Equation 13 (the corollary to Green's Theorem), which states

$$\oint_{\partial D} \mathbf{F} \cdot \hat{\mathbf{n}} ds = \iint_D \nabla \bullet F dA$$

we start with the one term in the identity that has a line integral. Namely,

$$\oint_{\partial D} f\nabla g \cdot \hat{\mathbf{n}} ds \overset{Eqn.13}{=} \iint_D \nabla \bullet (f\nabla g) dA$$

but now, recall on Quiz 9 #1(c) that $\nabla \bullet (f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f\nabla \cdot \mathbf{F}$. In particular, when $\mathbf{F} = \nabla g$, this gives us

$$\nabla \bullet (f\nabla g) = \nabla f \cdot \nabla g + f\nabla^2 g$$

where $\nabla^2 g = g_{xx} + g_{yy} + g_{zz}$. Hence, we have that

$$\oint_{\partial D} f\nabla g \cdot \hat{\mathbf{n}} ds \overset{Eqn.13}{=} \iint_D \nabla \bullet (f\nabla g) dA = \iint_D (\nabla f \cdot \nabla g + f\nabla^2 g) dA$$

in other words just writing it cleanly,

$$\oint_{\partial D} f\nabla g \cdot \hat{\mathbf{n}} ds = \iint_D (\nabla f \cdot \nabla g + f\nabla^2 g) dA$$

where now if we subtract the integral of $\nabla f \cdot \nabla g$, we have Green's first identity,

$$\oint_{\partial D} f\nabla g \cdot \hat{\mathbf{n}} ds - \iint_D \nabla f \cdot \nabla g dA = \iint_D f\nabla^2 g dA.$$

As mentioned in the review, this is kind of like a "2-D integration by parts" on $\iint_D f\nabla^2 g dA$. It's like "$u = f,\ dv = \nabla^2 g$" and the "evaluation at endpoints" of $uv\big|_a^b$ becomes the line integral over just the boundary, $\oint_{\partial D} f\nabla g \cdot \hat{n} ds$.

As you might guess, there is also a 3-D integration by parts type of formula, too.

#34

To begin this proof, with using Green's first identity (#33), apply it to both $\oint_C f\nabla g$ and $-\oint_C g\nabla f$ and we will result with the Left Hand Side because the $\iint_D \nabla f \cdot \nabla g dA$ terms subtract and cancel each other.

Mainly first rearrange Green's 1st identity to read as

$$\oint_{\partial D} f\nabla g \cdot \hat{\mathbf{n}} ds = \iint_D \nabla f \cdot \nabla g dA + \iint_D f\nabla^2 g dA.$$

Then, we have that for $\oint_C f\nabla g$ and $-\oint_C g\nabla f$, applying Green's 1st identity to each:

$$\oint_{\partial D} f\nabla g \cdot \hat{\mathbf{n}} ds = \iint_D \nabla f \cdot \nabla g dA + \iint_D f\nabla^2 g dA.$$

$$-\oint_{\partial D} g\nabla f \cdot \hat{\mathbf{n}} ds = -\iint_D \nabla f \cdot \nabla g dA - \iint_D g\nabla^2 f dA.$$

Adding the two, the middle integrals of $\nabla f \cdot \nabla g$ cancel, and we are left with

$$\oint_{\partial D} (f\nabla g - g\nabla f) \cdot \hat{\mathbf{n}} ds = \iint_D (f\nabla^2 g - g\nabla^2 f) dA.$$

#35

For this, to apply Green's first identity, the idea is that $\oint_C D_n g ds = \oint_C \nabla g \cdot \mathbf{n} ds$. The book's just trying to use new notation - don't get confused by it!

Mainly now, this is exactly in the form to use Green's first identity with $f = 1$, the constant function 1. With this, we see that the $\nabla f \cdot \nabla g$ integral is zero because f is constant, $\nabla f = \nabla(1) = 0$. The other integral of $f\nabla^2 g$ is also zero because g is harmonic, $\nabla^2 g = 0$ by definition.

$$\oint_{\partial D} 1\nabla g \cdot \hat{\mathbf{n}} ds \overset{\#33}{=} \iint_D 1\nabla^2 g dA + \iint_D \nabla(1) \cdot \nabla(g) dA$$

where the gradient of a constant is zero, and $\nabla^2 g = 0$ because g is harmonic,

$$= 0 + 0 = 0.$$

#36

With this proof, since we are looking at $|\nabla f|^2$, this is a hint that we should have $f = g$ because that way we get $\nabla f \cdot \nabla g = \nabla f \cdot \nabla f = |\nabla f|^2$. Mainly, this pops up in Green's first identity as the $\iint_D \nabla f \cdot \nabla g dA$ term. Now, using Green's identity, we can conclude this integral is zero because:

. Remember we have $f = g$ and f is harmonic so $\nabla^2 f = 0$ and the integral of $f\nabla^2 g$ is really now $f\nabla^2 f = 0$.

. Since $f = 0$ on the boundary, the line integral $\oint_C f\nabla f \cdot n ds$ is zero because C is exactly the boundary of the domain D.

$$\iint_D |\nabla f|^2 dA = \iint_D \nabla f \cdot \nabla f \ dA$$

where now applying #33 1st identity of Green, and rearranging terms,

$$= \oint_{\partial D} f(\nabla f) \cdot \hat{\mathbf{n}} ds - \iint_D f\nabla^2 f dA.$$

So, just rewriting, we have

$$\iint_D |\nabla f|^2 dA = \oint_{\partial D} f(\nabla f) \cdot \hat{\mathbf{n}} ds - \iint_D f\nabla^2 f dA.$$

The first integral of $f\nabla f$ is 0 because $f = 0$ on the boundary of D. The second integral of $f\nabla^2 f$ is 0 because f is harmonic, i.e. $\nabla^2 f = 0$, so then

$$\iint_D |\nabla f|^2 dA = 0 - 0 = 0.$$

16.6 Parametric Surfaces

51

First, always stick with definitions! Here,

$$Area(S) \overset{def}{=} \iint\limits_{x^2+y^2 \leq R^2} \sqrt{1 + f_x^2 + f_y^2} \ dxdy.$$

Since $|f_x|, |f_y| \leq 1$, we see that $\sqrt{1+f_x^2+f_y^2} \leq \sqrt{3}$ then. Hence,

$$Area(S) = \iint\limits_{x^2+y^2\leq R^2} \sqrt{1+f_x^2+f_y^2}\ dxdy$$

$$\leq \iint_{x^2+y^2\leq R^2} \sqrt{3}\ dxdy = \sqrt{3} \iint_{x^2+y^2\leq R^2} dA$$

where using the area interpretation of integrals,

$$\leq \sqrt{3} \cdot \pi R^2.$$

So, all we can conclude is that $Area(S)$ is $\leq \sqrt{3}\ \pi R^2$.

We can also note that $\sqrt{1+f_x^2+f_y^2} \geq 1$ so then in the same way, $Area(S) \geq \pi R^2$ too.

Hence, $\pi R^2 \leq Area(S) \leq \sqrt{3}\pi R^2$.

#58

I did this one in section.

#60 (and 59 is similar, I'll skip 59)

I dont think you'll need to know hyperbolic sine (sinh) and hyperbolic cosine (cosh)...

a) Here, we notice that aside from the scalings in front of the x, y, z parameterizations, we can make the trig identity $\sin^2 t + \cos^2 t = 1$ and also $\cosh^2 t - \sinh^2 t = 1$ work (I know hyperbolic trig functions are not that common, but they work similarly, though not exactly, like regular trig functions). To deal with the scalings, we just consider the quantities x/a, y/b, and z/c similar to what would be done in #59. Then, because we need $\cosh^2 - \sinh^2 = 1$, we expect to take

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \cosh^2 u \sin^2 v + \cosh^2 u \cos^2 v - \sinh^2 u$$

$$= \cosh^2 u(\sin^2 v + \cos^2 v) - \sinh^2 u = \cosh^2 v \cdot (1) - \sinh^2 u$$

$$= \cosh^2 u - \sinh^2 u = 1.$$

So, we see that the components of the surface parameterization satisfy

$$\frac{z^2}{c^2} - \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

which is the general equation of a hyperboloid of one sheet. Since the components satisfy the hyperboloid of one sheet's equation, it must be (part of) a one-sheet-hyperboloid.

b) It looks like a nuclear silo.

c) The first thing to note is that if $-3 \leq z \leq 3$, then with $c = 3$, we must have $-1 \leq \sinh u \leq 1$ which means that $-\sinh^{-1}(-1) < u < \sinh^{-1}(1)$. Secondly, $0 \leq v \leq 2\pi$ because this is what is essentially making x, y rotate around the graph (the nuclear silo).

Now, for the area formula, we need $r_u \times r_v$. Here, derivatives with cosh and sinh are a little different from their regular trig function counterparts,

$$\begin{cases} \mathbf{r_u} = < a\sinh u \cos v,\ b\sinh u \sin v,\ c\cosh u > \\ \mathbf{r_v} = < -a\cosh u \sin v,\ b\cosh u \cos v, 0 > \end{cases}$$

so then

$$\begin{aligned} \mathbf{r_u} \times \mathbf{r_v} &= < -bc\cosh^2 u \cos v,\ -ac\cosh^2 u \sin v,\ ab\sinh u \cosh u(\cos^2 v + \sin^2 v) > \\ &= < -bc\cosh^2 u \cos v,\ -ac\cosh^2 u \sin v,\ ab\sinh u \cosh u > \end{aligned}$$

and hence,

$$|\mathbf{r_u} \times \mathbf{r_v}| = \sqrt{b^2c^2\cosh^4 u \cos^2 v + a^2c^2\cosh^4 u \sin^2 v + a^2b^2\sinh^2 u \cosh^2 u}.$$

Then, for the area for part (b),

$$Area = \int_{v=0}^{2\pi} \int_{u=-\sinh^{-1}(-1)}^{\sinh^{-1}(1)} \sqrt{b^2c^2\cosh^4 u \cos^2 v + a^2c^2\cosh^4 u \sin^2 v + a^2b^2\sinh^2 u \cosh^2 u} dudv.$$

We can simplify this a little, and also plug in for $a = 1, b = 2, c = 3$ from part (b),

$$= \int_{v=0}^{2\pi} \int_{u=-\sinh^{-1}(-1)}^{\sinh^{-1}(1)} \cosh u \sqrt{36\cosh^2 u \cos^2 v + 9\cosh^2 u \sin^2 v + 4\sinh^2 u} dudv$$

Some observations, see # 61,63

Notice that #63 is on the practice final, as Problem 7.

For #61, first find the intersection of the paraboloid and sphere. I think it's just a circle. Then, the key is to get the right θ and in particular ϕ bounds by using some trig (inverse functions). The circle intersection should restrick the ϕ bound nicely between 0 and whatever ϕ-value the circle of intersection is at.

16.7 Surface Integrals

37

When our surface is the graph of $y = h(x, z)$, then we first parameterize our surface as

$$\mathbf{r}(x, z) = < x, h(x, z), z > \Longrightarrow \begin{cases} \mathbf{r_x} = < 1, h_x, 0 > \\ \mathbf{r_z} = < 0, h_z, 1 > . \end{cases}$$

Then, $\mathbf{r_x} \times \mathbf{r_z} = < h_x, -1, h_z >$. More importantly, this is the correct normal as the $\hat{j}$ component (in the y-direction) is -1. This means it goes to the left in the $-y$ direction.

Now, when we assume $\mathbf{F} = < P, Q, R >$, we have that

$$\iint_S \mathbf{F} \cdot \mathbf{dS} = \iint_{D_{x,z}} \mathbf{F}(x, h(x, z), z) \cdot (\mathbf{r_x} \times \mathbf{r_z}) dxdz$$

$$\iint_{D_{x,z}} < P(x, h(x, z), z), Q(x, h(x, z), z), R(x, h(x, z), z) > \cdot < h_x, -1, h_z > dxdz$$

$$= \iint_{D_{x,z}} \left(P\frac{\partial h}{\partial x} - Q + R\frac{\partial h}{\partial z} \right) dxdz.$$

38

This is basically identical to #37. There's one tiny, but important difference though! The normal needs to point in the positive $\hat{\imath}$ direction (in the $+x$ direction this time!).

Our surface is the graph of $x = k(y,z)$ now. We first parameterize our surface as

$$\mathbf{r}(y,z) = < k(y,z), y, z > \implies \begin{cases} \mathbf{r_y} = < k_y, 1, 0 > \\ \mathbf{r_z} = < k_z, 0, 1 > . \end{cases}$$

Then, $\mathbf{r_y} \times \mathbf{r_z} = < 1, -k_y, -k_z >$. More importantly, this is the correct normal as the $\hat{\jmath}$ component (in the x-direction) is $+1$. This means it goes towards us in the $+x$ direction.

Now, when we assume $\mathbf{F} = < P, Q, R >$, we have that

$$\iint_S \mathbf{F} \cdot \mathbf{dS} = \iint_{D_{y,z}} \mathbf{F}(x, h(x,z), z) \cdot (\mathbf{r_x} \times \mathbf{r_z}) dydz$$

$$\iint_{D_{y,z}} < P(k(y,z),y,z), Q(k(y,z),y,z), R(k(y,z),y,z) > \cdot < 1, -k_y, -k_z > dxdz$$

$$= \iint_{D_{y,z}} \left(P - Q\frac{\partial k}{\partial y} - R\frac{\partial k}{\partial z} \right) dydz.$$

48

This is exactly Problem 8(a) on the practice final.

49

We want to show that $\iint_S \mathbf{F} \cdot \mathbf{dS} = 0$ regardless of the radius of the sphere S. Here, we first write $\mathbf{F}$ using x, y, z variables as

$$\mathbf{F}(x,y,z) = \frac{c < x, y, z >}{(x^2+y^2+z^2)^{3/2}}.$$

Let us use that

$$\iint_S \mathbf{F} \cdot \mathbf{dS} = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS.$$

Here, recall from discussion that $\mathbf{n}_{sphere} = \nabla(x^2 + y^2 + z^2) = < 2x, 2y, 2z >$ so then when we normalize it, $\hat{\mathbf{n}} = \frac{<x,y,z>}{\sqrt{x^2+y^2+z^2}}$. Then,

$$\mathbf{F} \cdot \hat{\mathbf{n}} = \frac{c(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2} = \frac{c}{R^2}$$

because we are on the surface of the sphere S (here, R is the radius of S). Then,

$$\iint_S \mathbf{F} \cdot \mathbf{dS} = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \iint_S \frac{c}{R^2} \, dS$$

$$= \frac{c}{R^2} \iint_S dS = \frac{c}{R^2} \cdot \text{Surface Area}(S)$$

and using the surface area of a sphere,

$$= \frac{c}{R^2} \cdot 4\pi R^2 = 4\pi c.$$

This is a constant value, and more importantly independent of the radius. This verifies the claim that the flux of $\mathbf{F}$ across the sphere S is independent of the radius of S.

16.Review

#13 (#14 was on an old quiz I think..)

To show $\mathbf{F}$ is conservative, let's find f such that $\mathbf{F} = \nabla f$. Here, integrating the x-component,

$$f(x,y) = g(y) + \int (4x^3y^2 - 2xy^3)dx = g(y) + x^4y^2 - x^2y^3.$$

Then, $f_y = 2x^4y - 3x^2y^2 + 4y^3$ has to hold, it needs to equal the y component of $\mathbf{F}$.

$$f_y = g_y(y) + 2x^4y - 3x^2y^2 \quad \text{needs to equal} \quad 2x^4y - 3x^2y^2 + 4y^3$$

which means that

$$g_y(y) = 4y^3 \iff g(y) = y^4 + K$$

where K is a constant. Hence for $f(x,y) = y^4 + x^4y^2 - x^2y^3 + K$, we have that $\mathbf{F} = \nabla f$ so by the Fundamental theorem of line integrals,

$$\int_C \mathbf{F} \cdot \mathbf{dr} = f(final) - f(initial).$$

Here, the final point is $r(1) = (1+0, 2-1) \;=\; (1,1)$ and the initial point is $r(0) = \;(0,1)$. So,

$$\int_C \mathbf{F} \cdot \mathbf{dr} = f(1,1) - f(0,1) = (1+1-1+K) - (1+0-0+K) = 0.$$

(And we see, the constant doesn't affect the answer).

#21

Apply Green's theorem! We see that $Q = g(y)$ so then $\partial Q/\partial x = 0$. Similarly, $P = f(x)$ so $\partial P/\partial y = 0$ and hence by Green's this turns into an area integral of just 0, which is 0.

#22

First, start with that

$$\begin{aligned}
\nabla^2(fg) &= \frac{\partial^2}{\partial x^2}(fg) + \frac{\partial^2}{\partial y^2}(fg) + \frac{\partial^2}{\partial z^2}(fg) \\
&= \frac{\partial^2}{\partial x}(f_xg + fg_x) + \frac{\partial^2}{\partial y}(f_yg + fg_y) + \frac{\partial^2}{\partial z}(f_zg + fg_z) \\
&= [f_{xx}g + f_xg_x + f_xg_x + fg_{xx}] + [f_{yy}g + f_yg_y + f_yg_y + fg_{yy}] + [f_{zz}g + f_zg_z + f_zg_z + fg_{zz}] \\
&= [f_{xx} + f_{yy} + f_{zz}]g + 2[f_xg_x + f_yg_y + f_zg_z] + f[g_{xx} + g_{yy} + g_{zz}] \\
&= g\nabla^2 f + 2\nabla f \cdot \nabla g + f\nabla^2 g.
\end{aligned}$$

#23

The idea here is that applying Green's, we get that

$$\oint_{\partial D} f_y dx - f_x dy = \iint_D (-f_{xx} - f_{yy})dA = -\iint_D \nabla^2 f dA$$

and since f is harmonic, $\nabla^2 f = 0$,

$$= -\iint_D 0 dA = 0.$$

29

Only because this is one of the special cases, a sphere (or cylinder) where we know dS already. I'll do this in two ways, to hopefully clarify things.

1) Standard Way: We know what we parameterize the sphere as

$$\mathbf{r}(\theta,\phi) = < R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi > .$$

Then for the outward normal, we have to cross $\mathbf{r}_\phi$ and $\mathbf{r}_\theta$,

$$\begin{cases} \quad\quad\quad\quad\quad\quad \hat{i} \quad\quad\quad\quad\quad \hat{j} \quad\quad\quad\quad \hat{k} \\ \mathbf{r}_\phi = < R\cos\theta\cos\phi, R\sin\theta\cos\phi, -R\sin\phi > \\ \mathbf{r}_\theta = < -R\sin\theta\sin\phi, R\cos\theta\sin\phi, 0 > \end{cases}$$

$$\Longrightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta =< R^2\cos\theta\sin^2\phi, R^2\sin\theta\sin^2\phi, R^2\cos\phi\sin\phi(\cos^2\theta+\sin^2\theta) > .$$

So reducing with trig identities,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta =< R^2\cos\theta\sin^2\phi, R^2\sin\theta\sin^2\phi, R^2\cos\phi\sin\phi > .$$

(Indeed, this has outward orientation, all the components are positive). Then, since we integrate over the entire sphere, we have that after plugging in for x, y, z as functions of θ, ϕ,

$$\iint_S \mathbf{F}\cdot\mathbf{dS} = \int_0^{2\pi}\int_0^{\pi} \mathbf{F}(\mathbf{r}(\theta,\phi))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)d\phi d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} < R^2\cos\theta\sin\phi\cos\phi, -2R\sin\theta\sin\phi, 3R\cos\theta\sin\phi > \cdot < R^2\cos\theta\sin^2\phi, R^2\sin\theta\sin^2\phi, R^2\cos\phi\sin\phi >$$

$$= \int_0^{2\pi}\int_0^{\pi} [R^4\cos^2\theta\sin^3\phi\cos\phi - 2R^3\sin^2\theta\sin^3\phi + 3R^3\cos\theta\cos\phi\sin^2\phi] d\phi d\theta$$

the first and last terms go to zero because of $\cos\phi$ being odd about the $\phi = \pi/2$ axis, and the middle term gives, with $R = 2$ here for this sphere,

$$\int_0^{2\pi}\int_0^{\pi} -2^4\sin^2\theta\sin^3\phi\ d\phi d\theta = ... = -64\pi/3.$$

(The way to integrate this would be to use a half angle formula for $\sin^2\theta$ and then to rewrite $\sin^3\phi = \sin\phi(1-\cos^2\phi)$ and use a substitution $w = \cos\phi$.)

2) Shortcut Way: Since we know that $dS_{sphere} = R^2\sin\phi d\theta d\phi$, we can just find $\mathbf{F}\cdot\hat{\mathbf{n}}$ first. Here, we get $\mathbf{n}$ first by taking the gradient of the level surface that is the sphere, $x^2+y^2+z^2 = R^2$, in other words,

$$\mathbf{n} = \nabla(x^2+y^2+z^2) =< 2x, 2y, 2z > .$$

Then after we normalize it, we have

$$\hat{\mathbf{n}} = \frac{< 2x, 2y, 2z >}{\sqrt{4(x^2+y^2+z^2)}} = \frac{< x, y, z >}{\sqrt{x^2+y^2+z^2}} = \frac{1}{R} < x, y, z > \quad \text{on the sphere of radius } R.$$

So, taking dot product,

$$\mathbf{F}\cdot\hat{\mathbf{n}} = < xz, -2y, 3x > \cdot \frac{1}{R} < x, y, z >= \frac{1}{R}(x^2z - 2y^2 + 3xz).$$

Now we know that $x = R\cos\theta\sin\phi,\ y = R\sin\theta\sin\phi,\ z = R\cos\phi$ and also that $dS = R^2\sin\phi d\theta d\phi$ so

$$\iint_S \mathbf{F}\cdot\mathbf{dS} = \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{R}(R^3\cos^2\theta\sin^2\phi\cos\phi - 2R^2\sin^2\theta\sin^2\phi + 3R^2\cos\theta\sin\theta\sin\phi)\cdot R^2\sin\phi d\phi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} (R^4\cos^2\theta\sin^3\phi\cos\phi - 2R^3\sin^2\theta\sin^3\phi + 3R^3\cos\theta\sin\theta\sin^2\phi)d\phi d\theta$$

which is the same integral that resulted from the standard way, again, gets $-64\pi/3$.
(After some clever integration, and plugging in $R = 2$).

#30

A more standard example. First, we parameterize the paraboloid. Since z is between the paraboloid and the plane $z = 1$, we have that

$$\mathbf{r}(x, y) = < x, y, x^2 + z^2 >, \quad on \ \ x^2 + y^2 \leq 1.$$

Then, for examples, when we take the partials and cross them, we have

$$\mathbf{r}_y = < 0, 1, 2y >, \quad \mathbf{r}_x = < 1, 0, 2x >, \quad \Longrightarrow \mathbf{r_y}\times\mathbf{r_x} = < 2x, 2y, -1 > \quad (CAREFUL).$$

However, we want to have upward orientation, so the z-component of this normal vector must be positive, not negative (here, it is -1). Therefore, we should actually take

$$\mathbf{r_x}\times\mathbf{r_y} = < -2x, -2y, 1 >$$

for the correct orientation. Using this now, we have that by definition,

$$\iint_S \mathbf{F}\cdot\mathbf{dS} = \iint\limits_{x^2+y^2\leq 1} \mathbf{F}(\mathbf{r}(x, y))\cdot\mathbf{r_x}\times\mathbf{r_y}dxdy$$

$$= \iint\limits_{x^2+y^2\leq 1} < x^2, xy, x^2 + y^2 > \cdot < -2x, -2y, 1 > dxdy$$

$$= \iint\limits_{x^2+y^2\leq 1} (-2x^3 - 2xy^2 + x^2 + y^2)dxdy$$

$$= \iint\limits_{x^2+y^2\leq 1} [-2x(x^2 + y^2) + x^2 + y^2]dxdy$$

$$\overset{Polar}{=} \int_0^{2\pi}\int_0^1 (-2r\cos\theta\cdot r^2 + r^2)rdrd\theta = \int_0^{2\pi}\int_0^1 (-2r^4\cos\theta + r^3)drd\theta$$

in which the final answer is $\pi/2$.

#37

Don't fear, this is just fundamental theorem of line integrals! In general, if a curve looks really bad, it may be an implied hint to use the fundamental theorem.

First, we'd need to find f such that $\mathbf{F} = \nabla f$. You should get that $\mathbf{F} = \nabla f$ for the function $f(x,y,z) = x^3yz - 3xy + z^2 + K$. Then, by the fundamental theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(final) - f(initial) = f(0,3,0) - f(0,0,2) = 0 + K - 4 - K = -4.$$

If you need help to get f, here's the solution:
1) Integrate any component (I'll just pick the z component) and add in the *appropriate* constant function of integration to get an initial form of f.

$$f(x,y,z) = g(x,y) + \int (x^3y + 2z)dz = x^3yz + z^2 + g(x,y).$$

2) Solve for $g(x,y)$ by making f_x and f_y match the x and y components of the vector field $\mathbf{F}$.

$$f_x = 3x^2yz + g_x \overset{\text{needs to equal}}{\equiv} 3x^2yz - 3y.$$

Hence, $g_x = -3y$ so then $g(x,y) = h(y) + \int -3y dx = h(y) - 3xy$. So, from analyzing f_x, we see

$$f(x,y,z) = x^3yz + z^2 - 3xy + h(y).$$

$$f_y = x^3z - 3x + h_y(y) \overset{\text{needs to equal}}{\equiv} x^3z - 3x.$$

This means $h_y(y) = 0$ so thus $h(y) = K$ constant. We see

$$f(x,y,z) = x^3yz + z^2 - 3xy + K.$$